On non-archimedean $\Lambda(\alpha)$ – Gelfand-Philips Spaces

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Abstract

For a non-archimedean locally convex spaces, the $\Lambda(\alpha)$ – limited sets and operators are introduced and studied. We show that the finite product of $\Lambda(\alpha)$ – GP spaces is $\Lambda(\alpha)$ – GP space. We also show, under some conditions, that if $T_1, T_2, \ldots, T_n$ are any finite numbers of $\Lambda(\alpha)$ – limited operators from $E_i$ into $F_i$, then the operator $\prod_{i=1}^{n} T_i$ is $\Lambda(\alpha)$ – limited operator.

Key words: non-archimedean locally convex spaces, compactoid, limited sets, Kolmogorov diameter.

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INTRODUCTION:

In many branches of mathematics and its applications the valued fields of the real numbers \( \mathbb{R} \) and the complex numbers \( \mathbb{C} \) play a fundamental role. For quite some time one has been discussing the consequences of replacing in those theories \( \mathbb{R} \) or \( \mathbb{C} \) by the more general object of a non-archimedean valued field \( (K,|\cdot|) \).

In this paper we introduce \( \Lambda(\alpha) \)–compactoid sets and operators in non-archimedean locally convex spaces. We use the Kolmogorov diameters to show that the finite product of any \( \Lambda(\alpha) \)–compactoid sets is \( \Lambda(\alpha) \)–compactoid and to obtain for \( \Lambda(\alpha) \)–compactoid operators results resembling previously known properties of compact operators.

In the classical case of spaces over the real or complex field, analogous problems have been studied by several authors (see, for example, [3], [4], [7], [8]).

Preliminaries:

Let \( K \) be a field. A non-archimedean valuation on \( K \) is a function \( |\cdot|: K \to [0,\infty) \) such that for all \( \alpha, \beta \in K \) it satisfies:

\[
|\alpha| = 0 \text{ if and only if } \alpha = 0; \quad |\alpha \beta| = |\alpha| |\beta|; \quad \text{and } |\alpha + \beta| \leq \max\{|\alpha|, |\beta|\}.
\]

Let \( E \) be a linear space over the field \( K \). A non-archimedean seminorm on \( E \) is a seminorm which verifies the strong triangle inequality:

\[
\|x + y\| \leq \max\{\|x\|, \|y\|\} \quad \text{for all } x, y \in E.
\]

Throughout this paper \( K \) is a non-archimedean valued field that is complete with respect to the metric induced by the non-trivial valuation \( |\cdot| \), \( |K| \) is the set of all scalars \( |\lambda|, \lambda \in K \). Also \( E, F, \ldots \) are Hausdorff locally convex spaces over \( K \). We will denote by \( L(E,F) \) the vector space of all continuous linear operators from \( E \) into \( F \) and by \( cs(E) \) the collection of all continuous non-archimedean seminorm on \( E \). For \( p \in cs(E) \) and \( r > 0 \), \( B_p(0,r) \) will be the set \( \{x \in E : p(x) \leq r\} \). By [5] the collection of all neighborhoods \( B_p(0,1) (p \in cs(E)) \) is a local base of zero in \( E \).

By [1], a subset \( B \) of \( E \) is called compactoid if for every zero-neighbourhood \( U \) in \( E \) there exists a finite set \( S \subseteq E \) such that \( B \subseteq co(S) + U \), where \( co(S) \) is the absolutely convex hull of \( S \).
Let us denote by the \( c_0 \) the space of all sequences in \( K \) converging to zero and by \( l_\infty(K) \) the space of all bounded sequences in \( K \).

**\( \Lambda(\alpha) \) – compactoid sets**

Let \( \alpha = (\alpha_n)_{n=1}^\infty \) be an increasing sequence of non-negative real numbers tending to infinity and satisfying
\[
\alpha_n \leq c \alpha_n \quad \text{for some } c > 0 \quad \text{and all } n \in \mathbb{N}.
\]

For \( R \in \mathbb{N} \setminus \{0\} \), \( R \geq 1 \) and a sequence \( \xi = (\xi_n)_{n=1}^\infty \) in \( K \), we define
\[
p_R(\xi) = \sup_n R^{\alpha_n} |\xi_n|.
\]

The non-archimedean power series space \( \Lambda(\alpha) \) is the space of all sequences \( \xi \) in \( K \) for which \( p_R(\xi) < \infty \) for all \( R \in \mathbb{N} \setminus \{0\} \), \( R \geq 1 \). On \( \Lambda(\alpha) \) we consider the locally convex topology generated by the family \( \{p_R : R \in \mathbb{N} \setminus \{0\}, R \geq 1\} \) of non-archimedean seminorms (Compare [2], [6]). Under this topology \( \Lambda(\alpha) \) is a complete Hausdorff locally convex space over \( K \), and condition (1) is equivalent to \( \Lambda(\alpha) \times \Lambda(\alpha) \cong \Lambda(\alpha) \) and called stability (see [6]).

In case \( \alpha = (1, 1, \ldots, 1) \), we have \( \Lambda(\alpha) = l_\infty(K) \) and in case \( \alpha = (\log(n + 1)) \) we obtain the space of all rapidly decreasing sequences [8].

By \([x] \) we mean the integer part of the real number \( x \) such that \([x] = \eta \) if \( x = n + \mu, \ 0 \leq \mu < 1 \).

**Proposition 2.1.**

If \( \xi = (\xi_0, \xi_1, \xi_2, \ldots) \in \Lambda(\alpha) \), then \((\xi_{[\frac{n}{2}]}_{n=0}^\infty) = (\xi_0, \xi_1, \xi_1, \xi_1, \ldots) \in \Lambda(\alpha) \).

**Proof.** Let \( \xi \in \Lambda(\alpha) \). Then for \( R \in \mathbb{N} \setminus \{0\} \), \( R \geq 1 \), and \( n > 1 \) we have
\[
R^{\alpha_{[\frac{n}{2}]}(\xi_0, \xi_1, \xi_2)} \leq R^{\alpha_{[\frac{n}{2}]}(\xi_0, \xi_1, \xi_2)} \leq R^{\alpha_{[\frac{n}{2}]}(\xi_1, \xi_2)} \leq R^{\alpha_{[\frac{n}{2}]}(\xi_2)} \leq R^{\alpha_{[\frac{n}{2}]}(\xi_n)} < \infty,
\]

And so \((\xi_{[\frac{n}{2}]}_{n=0}^\infty) \in \Lambda(\alpha) \).

For a bounded subset \( B \) of a locally convex space \( E \) over \( K \), a \( p \in cs(E) \) and a non-negative integer \( n \), the \( n \)th Kolmograve diameter \( \delta_{n,p}(B) \) of \( B \) with respect to \( p \) is the infimum of all \( |\mu| \), \( \mu \in K \), for which there exists a subspace \( F \) of \( E \) with \( \dim(F) \leq n \), such that \( A \subset F + \mu B_p(0,1) \) (see [6]). These \( n \)th Kolmograve diameters satisfy the following properties:
Proposition 2.2.
(Compare [3], p16, [7], p208, [8]):
(i) \( \delta_{0,p}(B) \geq \delta_{1,p}(B) \geq \delta_{2,p}(B) \geq \ldots \geq 0 \) for all \( p \in cs(E) \).
(ii) If \( B_i \subseteq B \) and \( p \leq q \), then \( \delta_{n,p}(B_i) \leq \delta_{n,q}(B) \).
(iii) If \( T \in L(E,F) \), then for all \( p \in cs(F) \) there exists \( q \in cs(E) \) such that \( \delta_{n,p}(T(B)) \leq \delta_{n,q}(B) \).

Definition 2.3.
(Compare [6]) A subset \( B \) of \( E \) is called \( \Lambda(\alpha) \) compactoid if for all \( p \in cs(E) \) there exists \( \xi = (\xi_n)_{n=0}^{\infty} \in \Lambda(\alpha) \) such that \( \delta_{n,p}(B) \leq |\xi_n| \) for all \( n \) (or equivalently \( \sup_{n} R^a \cdot \delta_{n,p}(B) < \infty \) for all \( R \in |K|/\{0\}, R \geq 1 \)).

In case, \( \Lambda(\alpha) = c_0 \), the concept of \( \Lambda(\alpha) \) compactoid set coincide with the concept of a compatoid set (see [6]).

Proposition 2.4
Let \( E = \prod_{i=1}^{n} E_i \), where each \( E_i \) is a locally convex space whose topology generated by the family \( cs(E_i) \) of non-archimedean seminorms, and let \( B_i \) be any bounded subset of \( E_i \), then for all \( p \in cs(E) \) there exist \( p_i \in cs(E_i) \), \( i = 1,2,\ldots,n \), such that
\[
\delta_{s,p}(\prod_{i=1}^{n} B_i) \leq \inf \max \{\max(\delta_{k_i,p_i}(B_i)) \} = \prod_{i=1}^{n} B_i \subset U_i \] for each \( U_i \) is a neighborhood of zero in \( E_i \).

Proof: Suppose \( p \in cs(E) \), then the neighborhood \( B_p(0,1) \) of zero in \( E \) can be taken in the form \( U = \prod_{i=1}^{n} U_i \), where \( U_i \) is a neighborhood of zero in \( E_i \).

Since the collection \( \{B_p(0,1) : p \in cs(E_i)\} \) is a local base of zero in \( E_i \), then for each \( U_i \), there exists \( p_i \in cs(E_i) \) such that \( B_{p_i}(0,1) \subseteq U_i \), so \( \prod_{i=1}^{n} B_{p_i}(0,1) \). Now, according to the definitions of \( \delta_{k_i,p_i}(B_i) \), \( i = 1,2,\ldots,n \), we have for \( \varepsilon > 0 \), there exist subspaces \( F_i, \ldots,F_n \) of \( E_1,\ldots,E_n \), respectively, with \( \dim(F_i) \leq k_i, \ldots, \dim(F_n) \leq k_n \), and \( \mu_1,\ldots,\mu_n \in K \) such that \( |\mu_i| \leq \delta_{k_i,p_i}(B_i) + \varepsilon \) and \( B_i \subseteq \mu_i B_{p_i}(0,1) + F_i \). This implies that
\[
\prod_{i=1}^{k} B_i \subset \prod_{i=1}^{n} \mu B_{\mu_i}(0,1) + \prod_{i=1}^{n} F_i \subset \mu B_{\mu}(0,1) + \prod_{i=1}^{n} F_i
\]

where \(|\mu| = \max(\{|\mu_1|, \ldots, |\mu_n|\})\). Since \(\dim(\prod_{i=1}^{n} F_i) \leq k_1 + k_2 + \ldots + k_n \leq s\), then
\[
\delta_{s,\mu}(\prod_{i=1}^{n} B_i) \leq |\mu| = \max(\{|\mu_1|, \ldots, |\mu_n|\}) \leq (\max(\delta_{k_1,\mu_1}(B_1), \ldots, \delta_{k_n,\mu_n}(B_n))) + \varepsilon.
\]

Since \(\varepsilon > 0\) was arbitrary, we get
\[
\delta_{s,\mu}(\prod_{i=1}^{n} B_i) \leq \max(\delta_{k_1,\mu_1}(B_1), \ldots, \delta_{k_n,\mu_n}(B_n)),
\]

and since this estimation is true for any choice of \(k_1 + k_2 + \ldots + k_n \leq s\), then
\[
\delta_{s,\mu}(\prod_{i=1}^{n} B_i) \leq \min\{\max(\delta_{k_1,\mu_1}(B_1), \ldots, \delta_{k_n,\mu_n}(B_n))\}
\]

**Corollary 2.5**
\[
\delta_{s,\mu}(\prod_{i=1}^{n} B_i) \leq \max(\delta_{\left\lfloor \frac{s}{2^r} \right\rfloor,\mu_1}(B_1), \ldots, \delta_{\left\lfloor \frac{s}{2^r} \right\rfloor,\mu_n}(B_n))
\]

where \(r\) is the smallest integer such that \(n < 2^r\).

**Proof:** Since \(\sum_{i=1}^{n} \frac{S}{2^r} \leq s\), it follows from proposition (2.4) that
\[
\delta_{s,\mu}(\prod_{i=1}^{n} B_i) \leq \min\{\max(\delta_{k_1,\mu_1}(B_1), \ldots, \delta_{k_n,\mu_n}(B_n))\}
\]
\[
\leq \max(\delta_{\left\lfloor \frac{s}{2^r} \right\rfloor,\mu_1}(B_1), \ldots, \delta_{\left\lfloor \frac{s}{2^r} \right\rfloor,\mu_n}(B_n)).
\]

**Theorem 2.6.**
The product of any finite number of \(\Lambda(\alpha)\)–compactoid sets is \(\Lambda(\alpha)\)–compactoid.

**Proof.** Suppose \(\prod_{i=1}^{n} B_i \subset \prod_{i=1}^{n} E_i\), where each \(B_i\) is a \(\Lambda(\alpha)\)–compactoid in \(E_i\). Clearly \((\delta_{s,\mu_1}(B_1))_{s=0}^{\infty} \in \Lambda(\alpha)\) for all \(\mu_i \in \text{cs}(E_i)\), it follows from proposition (2.1) that \((\delta_{\left\lfloor \frac{s}{2^r} \right\rfloor,\mu_1}(B_1))_{s=0}^{\infty} \in \Lambda(\alpha)\) for all \(\mu_i \in \text{cs}(E_i)\), where \(r\) is
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the smallest integer such that $n < 2^r$. Now from corollary (2.5), for all $p \in cs(E)$ there exist $p_i \in cs(E_i)$, $i = 1,2,\ldots,n$, such that

$$\delta_{p_i} \left( \prod_{i=1}^{n} B_i \right) \leq \max \left\{ \frac{s}{2}, p_i \right\} \left( B_i \right) \leq \delta_{p_i} \left( B_i \right) \leq \delta_{p_i} \left( B_i \right) + \sum_{s=0}^{n} \delta_{p_i} \left( B_i \right) \forall s = 0,1,\ldots$$

Thus, $(\delta_{p_i} \left( \prod_{i=1}^{n} B_i \right))_{s=0}^{\infty} \in \Lambda(\alpha)$ for all $p \in cs(E)$, which prove that $\prod_{i=1}^{n} B_i$ is $\Lambda(\alpha)$-compactoid in $E$.

**Theorem 2.7**
The continuous linear image of any $\Lambda(\alpha)$-compactoid set is $\Lambda(\alpha)$-compactoid.

**Proof.** Let $T \in L(E,F)$ and let $B$ be a subset of $E$ which is $\Lambda(\alpha)$-compactoid, then $(\delta_{p_i} (B))_{s=0}^{\infty} \in \Lambda(\alpha)$ for all $p \in cs(E)$. Now, by property (iii) of proposition (2.2), for each $q \in cs(F)$ there exists $p \in cs(E)$ such that $\delta_{n,q} (T(B)) \leq \delta_{n,p} (B)$ for all $n \in N$, and so $T(B)$ is $\Lambda(\alpha)$-compactoid in $F$.

**Corollary 2.8**
If $B = \prod_{i=1}^{n} B_i$ is $\Lambda(\alpha)$-compactoid subset of a locally convex space $E = \prod_{i=1}^{n} E_i$, then $B_i$ is $\Lambda(\alpha)$-compactoid.

**Proof:** It follows from the fact that, the projection operator $P_i$ from $\prod_{i=1}^{n} E_i$ into $E_i$ is continuous and $P_i(\prod_{i=1}^{n} B_i) = B_i$.

### 3 $\Lambda(\alpha)$-Limited sets and $\Lambda(\alpha)$-GP-spaces.

**Definition 3.1.**
(Compare [6]) An operator $T \in L(E,F)$ between two locally convex spaces $E, F$ over $K$ is called $\Lambda(\alpha)$-compactoid if there exists a neighborhood $V$ of zero in $E$ such that $T(V)$ is $\Lambda(\alpha)$-compactoid in $F$. 

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In case, $\Lambda(\alpha) = c_0$ the concept of $\Lambda(\alpha)$–compactoid operator coincide with the concept of a compact operator (see [6]).

**Definition 3.2**

Compare [1]

i) A bounded subset $B$ of $E$ is called $\Lambda(\alpha)$–limited in $E$ if and only if for each continuous linear map $T$ from $E$ to $c_0$, $T(B)$ is $\Lambda(\alpha)$–compactoid in $c_0$.

ii) An operator $T \in L(E, F)$ is called $\Lambda(\alpha)$–limited if there exists a zero-neighborhood $U$ in $E$ such that $T(U)$ is $\Lambda(\alpha)$–limited in $F$.

**Proposition 3.3.**

i) Every $\Lambda(\alpha)$–compactoid subset of $E$ is $\Lambda(\alpha)$–limited in $E$.

ii) If $B$ is $\Lambda(\alpha)$–limited in $E$ and $T \in L(E, F)$, then $T(B)$ is $\Lambda(\alpha)$–limited in $F$ where $L(E, F)$ denotes the vector space of all continuous linear maps from $E$ to $F$.

iii) If $B$ is $\Lambda(\alpha)$–limited in $E$ and $D \subset B$, then $D$ is $\Lambda(\alpha)$–limited in $E$.

iv) Let $M$ be a subspace of $E$ and $B \subset M$. If $B$ is $\Lambda(\alpha)$–limited in $M$ then $B$ is $\Lambda(\alpha)$–limited in $E$.

**Proof:**

i) Let $B$ be any $\Lambda(\alpha)$–compactoid subset of $E$ and let $T \in L(E, c_0)$. It follows from property (iii) of proposition (2.2) that for all $p \in cs(F)$ there exists $q \in cs(E)$ such that $\delta_{n,p}(T(B)) \leq \delta_{n,q}(B)$ and so $T(B)$ is $\Lambda(\alpha)$–compactoid in $c_0$. Therefore $B$ is $\Lambda(\alpha)$–limited in $E$.

ii) Suppose $B$ is $\Lambda(\alpha)$–limited in $E$ and $T \in L(E, F)$. Let $G \in L(F, c_0)$, then $G \circ T \in L(E, c_0)$, so $G(T(B))$ is $\Lambda(\alpha)$–compactoid in $c_0$. Therefore $T(B)$ is $\Lambda(\alpha)$–limited in $E$.

iii) Let $D \subset B$ and $T \in L(E, F)$, then $T(D) \subset T(B)$, and hence, by property (ii) of proposition (2.2), $\delta_{n,p}(T(D)) \leq \delta_{n,p}(B)$ for all $p \in cs(F)$, and so $T(D)$ is $\Lambda(\alpha)$–compactoid in $c_0$. And this complete the proof.

iv) Let $M$ be a subspace of $E$ and let $B \subset M$ be $\Lambda(\alpha)$–limited in $M$. If $T \in L(E, c_0)$, then the restriction operator $T|_{M} \in L(M, c_0)$ is continuous, and so $T|_{M}(B) = T(B)$ is $\Lambda(\alpha)$–compactoid in $c_0$, Thus $B$ is $\Lambda(\alpha)$–limited in $E$.

**Definition 3.4**

Compare [1]
A locally convex space $E$ is called $\Lambda(\alpha)$–Gelfand-Philips space ($\Lambda(\alpha)$–GP-space in short) if every $\Lambda(\alpha)$–limited set in $E$ is $\Lambda(\alpha)$–compactoid.

**Theorem 3.5**

If $\prod_{i=1}^{n} B_i \subseteq \prod_{i=1}^{n} E_i$ is $\Lambda(\alpha)$–limited, then $B_i$ is $\Lambda(\alpha)$–limited.

**Proof:** Let $T : E_i \to c_0$ be any continuous linear operator, then $P_i : \prod E_i \to E_i$ is continuous, then $T \circ P_i : \prod E_i \to c_0$ is continuous, so

$$T \circ P_i(\prod B_i) = T(B_i)$$

is $\Lambda(\alpha)$–compactoid, hence $B_i$ is $\Lambda(\alpha)$–limited.

**Proposition 3.6**

i) A subspace of $\Lambda(\alpha)$–GP-space is $\Lambda(\alpha)$–GP-space.

ii) The product of a any finite numbers of $\Lambda(\alpha)$–GP-spaces is $\Lambda(\alpha)$–GP-space.

**Proof:**

i) Let $M$ be a subspace of $\Lambda(\alpha)$–GP-space $E$ and let $B_i$ be any $\Lambda(\alpha)$–limited $M$, then, by (iv) of proposition (3.3), $B_i$ is $\Lambda(\alpha)$–limited in $E$. Since $E$ is $\Lambda(\alpha)$–GP-space, then $B_i$ is $\Lambda(\alpha)$–compactoid in $E$, and hence in $M$, and therefore $M$ is $\Lambda(\alpha)$–GP-space.

ii) Let $E_1, E_2, \ldots, E_n$ be a finite numbers of $\Lambda(\alpha)$–GP-spaces and let $B = \prod_{i=1}^{n} B_i \subseteq \prod_{i=1}^{n} E_i$ be $\Lambda(\alpha)$–limited set in $\prod_{i=1}^{n} E_i$. It follows from theorem (3.5) that $B_i$ is $\Lambda(\alpha)$–limited in $E_i$. Since each $E_i$ is $\Lambda(\alpha)$–GP-space, then $B_i$ is $\Lambda(\alpha)$–compactoid in $E_i$. Now by theorem (2.6) $\prod_{i=1}^{n} B_i$ is $\Lambda(\alpha)$–compactoid in $E$. So $\prod_{i=1}^{n} E_i$ is $\Lambda(\alpha)$–GP-space.

**Definition 3.7**

Let $T_1, T_2, \ldots, T_n$ be any finite numbers of continuous linear operators from $E_i$ into $F_j$, then The operator $\prod_{i=1}^{n} T_i : \prod_{i=1}^{n} E_i \to \prod_{i=1}^{n} F_j$ is defined by
\[ \prod_{i=1}^{n} T_i(x_1, x_2, \ldots, x_n) = (T_1 x_1, T_2 x_2, \ldots, T_n x_n). \]

**Proposition 3.8**

Let \( T_1, T_n, \ldots, T_n \) be any finite numbers of \( \Lambda(\alpha) \)-limited operators from \( E_i \) into \( F_i \), where \( F_1, F_n, \ldots, F_n \) are \( \Lambda(\alpha) \)-GP spaces, then the operator \( \prod_{i=1}^{n} T_i \) is \( \Lambda(\alpha) \)-limited operator.

**Proof:** Suppose \( T_1, T_n, \ldots, T_n \) are \( \Lambda(\alpha) \)-limited operators from \( E_i \) into \( F_i \), then there exist neighborhoods \( V_1, V_n, \ldots, V_n \) of zero in \( E_1, E_n, \ldots, E_n \) such that \( T_1(V_1), T_2(V_2), \ldots, T_n(V_n) \) are \( \Lambda(\alpha) \)-limited sets in \( F_1, F_2, \ldots, F_n \). Since \( F_1, F_2, \ldots, F_n \) are \( \Lambda(\alpha) \)-GP spaces, then \( T_1(V_1), T_2(V_2), \ldots, T_n(V_n) \) are \( \Lambda(\alpha) \)-compactoid sets, and so by theorem (2.6) and proposition (3.3, (i)) it follows that \( \prod_{i=1}^{n} T_i(V_i) \) is \( \Lambda(\alpha) \)-limited set in \( \prod_{i=1}^{n} F_i \). Thus \( \prod_{i=1}^{n} T_i \) is \( \Lambda(\alpha) \)-limited operator.
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